

APPROXIMATE GROUPS, II: THE SOLVABLE LINEAR CASE

EMMANUEL BREUILLARD AND BEN GREEN

ABSTRACT. We describe the structure of “ K -approximate subgroups” of solvable subgroups of $\mathrm{GL}_n(\mathbb{C})$, showing that they have a large nilpotent piece. By combining this with the main result of our recent paper on approximate subgroups of torsion-free nilpotent groups [3], we show that such approximate subgroups are efficiently controlled by nilpotent progressions.

CONTENTS

1. Introduction	1
2. Preliminaries from multiplicative combinatorics	3
3. Proof of the main theorems	5
References	10

1. INTRODUCTION

This paper is the second in a series concerning *approximate groups*. The first paper in the series is [3], which discusses approximate subgroups of torsion-free nilpotent groups. The reader is referred to that paper for a more extensive discussion on the background to the material we discuss here. Let us recall the definition of “ K -approximate group”.

Definition 1.1 (Approximate groups). Let G be some group and let $K \geq 1$. A set $A \subseteq G$ is called a K -approximate group if

- (i) It is symmetric, i.e. if $a \in A$ then $a^{-1} \in A$, and the identity lies in A ;
- (ii) There is a symmetric subset X lying in $A \cdot A$ with $|X| \leq K$ such that $A \cdot A \subseteq X \cdot A$.

Let us also recall Tao’s notion of *control* [17].

Definition 1.2 (Control). Suppose that A and B are two sets in some ambient group, and that $K \geq 1$ is a parameter. We say that A is K -controlled by B , or that B K -controls A , if $|B| \leq K|A|$ and there is some set X in the ambient group with $|X| \leq K$ and such that $A \subseteq (X \cdot B) \cap (B \cdot X)$.

SOLVABLE LINEAR GROUPS. Our aim in this paper is to study K -approximate subgroups of solvable subgroups of $\mathrm{GL}_n(\mathbb{C})$, for a fixed integer n . Recall that these are

groups G such that the derived series

$$G_{(0)} = G, G_{(1)} = [G_{(0)}, G_{(0)}], G_{(2)} = [G_{(1)}, G_{(1)}], \dots$$

terminates with $G_{(s+1)} = \{\text{id}\}$. For example the group $G = \begin{pmatrix} \mathbb{C}^\times & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C}^\times & \mathbb{C} \\ 0 & 0 & \mathbb{C}^\times \end{pmatrix}$ is solvable, as is any subgroup of it.

This example is actually fairly typical, as the following well-known theorem of Mal'cev [11] (see also [19, Theorem 3.6]) shows. In this result and the rest of the paper we write $\text{Upp}_n(\mathbb{C})$ for the (solvable) subgroup of $\text{GL}_n(\mathbb{C})$ consisting of upper-triangular matrices.

Theorem 1.3 (Mal'cev). *Suppose that G is a solvable subgroup of $\text{GL}_n(\mathbb{C})$. Then there is a normal subgroup $H \triangleleft G$ whose index in G is bounded by some function $F(n)$ of n only and which is conjugate to a subgroup of $\text{Upp}_n(\mathbb{C})$.*

We cannot take $F(n) = 1$, that is to say not all solvable subgroups are conjugate to a group of upper triangular matrices. Indeed any non-abelian finite group has an irreducible linear representation of dimension at least 2.

We now state our main results.

Theorem 1.4. *Let $K \geq 1$ be a parameter, suppose that $G \subseteq \text{GL}_n(\mathbb{C})$ is a solvable group, and suppose that $A \subseteq G$ is a K -approximate group. Then A is K^{C_n} -controlled by a K^{C_n} -approximate group $B \subseteq G$ which generates a nilpotent group of step at most $n - 1$ contained in a conjugate of $\text{Upp}_n(\mathbb{C})$.*

Remarks. We recalled in [3] that all finitely-generated torsion-free nilpotent groups may be realised as linear groups. The same is by no means true for solvable groups; indeed if a group G embeds in $\text{Upp}_n(\mathbb{C})$ then $[G, G]$ is nilpotent, a very special property in the class of all solvable groups. The paper of Tao [17] applies to solvable groups in general, and so is much broader in scope than our work. That paper comes with extremely weak bounds, however, whereas we obtain polynomial dependence on the approximation parameter K . The work of Sanders [13] also applies to a more general class of group than ours. His bounds are in a sense better than Tao's, but his structural conclusions are somewhat weaker.

Corollary 1.5. *Let $K \geq 1$ be a parameter, suppose that $G \subseteq \text{GL}_n(\mathbb{C})$ is a solvable group, and suppose that $A \subseteq G$ is a K -approximate group. Then A is $e^{K^{C_n}}$ -controlled by a nilpotent progression of dimension at most K^{C_n} and step at most $n - 1$.*

Remarks. Let us make particular note at this point of our dependence on Part I of the series [3], where the concept of a nilpotent progression that we have invoked here is introduced and developed at some length.

2. PRELIMINARIES FROM MULTIPLICATIVE COMBINATORICS

Let us begin by stating the basic properties of approximate groups. The next proposition was also stated in [3] as Proposition 2.1. It is due to Tao [16].

Proposition 2.1 (Approximate groups and control). *Let $K \geq 1$ be a parameter and let A be a set in some ambient group G . If $n \geq 1$ is an integer we write $A^n = \{a_1 \dots a_n : a_1, \dots, a_n \in A\}$ and $A^{\pm n} = \{a_1^{\varepsilon_1} \dots a_n^{\varepsilon_n} : a_1, \dots, a_n \in A, \varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}\}$.*

- (i) *If $\pi : G \rightarrow H$ is a homomorphism and if $A \subseteq G$ is a K -approximate group then $\pi(A)$ is a K -approximate subgroup of H .*
- (ii) *If A is a K -approximate group then $|A^{\pm n}| = |A^n| \leq K^{n-1}|A|$ and A^n is K^{n+1} -controlled by A .*
- (iii) *If B, C are further subsets of G and if A is K -controlled by B and B is K -controlled by C , then A is K^2 -controlled by C .*
- (iv) *If the doubling constant $|A^2|/|A|$ is at most K then there is an $f_1(K)$ -approximate group $B \subseteq A^{\pm 3}$ which $f_2(K)$ -controls A . If the tripling constant $|A^3|/|A|$ is at most K then we may take $B = A^{\pm 3}$.*
- (v) *If A is a K -approximate group and if $A' \subseteq A$ is a subset with $|A'| \geq |A|/K$ then $A'^{\pm 3}$ is an $f_3(K)$ -approximate group which $f_4(K)$ -controls A . The same is in fact true under the essentially weaker assumption that $|A^3| \leq K|A|$.*

All of the quantities $f_1(K), \dots, f_4(K)$ can be taken to be polynomial in K .

Let us recall also the following simple but surprisingly powerful observation, which is essentially [16, Lemma 3.6] phrased using the language of control.

Lemma 2.2 (Nonabelian Ruzsa covering lemma). *Suppose that A and B are finite sets in some ambient group such that B is symmetric and for which $|A \cdot B|, |B \cdot A| \leq K|B|$ and $|B^2| \leq K|A|$. Then A is $2K$ -controlled by B^2 .*

Proof. Take X_1 to be a maximal subset of A with the property that the translates Bx , $x \in X$, are all disjoint. It is easy to see that $|X_1| \leq K$ and that $A \subseteq B^{-1} \cdot B \cdot X_1 = B^2 \cdot X_1$. Similarly there is a set X_2 with $|X_2| \leq K$ such that $A \subseteq X_2 \cdot B^2$. Taking $X := X_1 \cup X_2$ we see that $A \subseteq (B^2 \cdot X) \cap (X \cdot B^2)$, as required. \square

We supplement these last two results with the following lemma, which allows one to pass to finite index subgroups when describing the structure of approximate groups. In view of Mal'cev's theorem it allows us to reduce the proof of Theorem 1.4 to the case of approximate subgroups of $\text{Upp}_n(\mathbb{C})$.

Lemma 2.3. *Suppose that G is a group and that $H \leq G$ is a subgroup of G with finite index r . Suppose that A is a K -approximate subgroup of G . Then there is a $(rK)^C$ -approximate subgroup $S \subseteq H$ which $(rK)^C$ -controls A .*

Proof. By the pigeonhole principle there is a coset xH of H such that $|A \cap xH| \geq |A|/r$. Writing $A' := A \cap xH$ and $B := A'^{-1}A'$, we have $B \subseteq H \cap A^2$. By Lemma 2.2 and Proposition 2.1 (ii) we see that A is $(rK)^C$ -controlled by B^2 . By Proposition 2.1 (ii) and (iv) there is a $(rK)^C$ -approximate group $S \subseteq B^{\pm 6} \subseteq H$ which $(rK)^C$ -controls B^2 . By the transitivity of the notion of control, S also $(rK)^C$ -controls A . \square

We saw in Proposition 2.1 that a K -approximate group A satisfies the tripling condition $|A^3| \leq K^2|A|$. However there are examples of sets with small tripling which are not approximate groups (an example is a random subset of $\{1, \dots, N\}$ of size $N/2$). For the purposes of this paper it turns out to be convenient to work with the small tripling condition rather than the approximate group condition. In the remainder of this section we gather some preliminaries concerning sets with small tripling (or even doubling).

Lemma 2.4 (Iterated products). *Suppose that A is a subset of some ambient group such that $|A^3| \leq K|A|$. Then we have $|A^{\pm n}| \leq K^{C_n}|A|$.*

Proof. This is not too hard to establish by repeated applications of the Ruzsa triangle inequality. The details may be found in [16, Lemma 3.4]. \square

Lemma 2.5 (Almost uniform fibres). *Suppose that G and H are groups and that $\pi : G \rightarrow H$ is a homomorphism. Suppose that $A \subseteq G$ is a set with $|A^2| \leq K|A|$. For each $x \in \pi(A)$ write A_x for the fibre $\pi^{-1}(x) \cap A$. Then $\max_{x \in \pi(A)} |A_x| \leq K \min_{x \in \pi(A)} |A_x|$.*

Proof. Let $S \subseteq A$ be a set containing precisely one element in each fibre A_x . Manifestly

$$|S| \min_{x \in \pi(A)} |A_x| \leq |A|.$$

On the other hand for any $x \in \pi(A)$ the set $A_x \cdot S$ has size $|A_x||S|$ and lies in $A \cdot A$, a set of size at most $K|A|$ by the definition of what it means for A to be a K -approximate group. Therefore

$$|S| \max_{x \in \pi(A)} |A_x| \leq K|A|.$$

Comparing these two inequalities yields the result. \square

A fairly quick corollary of this is the next result, which examines the extent to which small tripling is preserved under homomorphisms. Results very similar to this one may be found in Section 7 of [16] and also in Lemma 7.3 of [9].

Corollary 2.6 (Small tripling preserved under homomorphisms). *Suppose that G and H are groups and that $\pi : G \rightarrow H$ is a homomorphism. Suppose that $A \subseteq G$ is a set with $|A^3| \leq K|A|$. Then $|\pi(A)^3| \leq K^C|\pi(A)|$.*

Proof. Write \approx to denote two quantities which are equal up to some fixed power of K . We have $|A_x| \approx |A|/|\pi(A)|$ and $|(A^3)_x| \approx |A^3|/|\pi(A)^3|$ for all x by the previous lemma

and Lemma 2.4 (note that $\pi(A^3) = \pi(A)^3$, since A is a homomorphism). Now it is clear that A^3 contains a translate of every fibre of A , and so $\max_x |(A^3)_x| \geq \max_x |A_x|$. Combining these facts with the assumption that $|A^3| \approx |A|$ gives $|\pi(A)^3| \approx |\pi(A)|$, as required. \square

The behaviour of sets with small tripling under intersection with a subgroup $H \leq G$ is a little more subtle. Indeed suppose that G is large but finite and that H is a small subgroup. If $S \subseteq H$ is any set and if $A := S \cup (G \setminus H)$ then A will have tripling very close to 1 yet $S = A \cap H$ need not have any special structure. This phenomenon is not particularly persistent, however, as the following lemma shows. Lemmas almost identical to this one may be found in [9] and [17].

Lemma 2.7 (Sumsets and intersections). *Let G be any group. Suppose that $A \subseteq G$ is a finite symmetric set containing the identity with $|A^3| \leq K|A|$. Let $H \leq G$ be a subgroup. Then for each $n \geq 2$ we have*

$$|A^2 \cap H| \leq |A^n \cap H| \leq K^{C_n} |A^2 \cap H|.$$

Proof. Since A contains the identity the lower bound is obvious. To prove the upper bound list the right cosets of H which have nonempty intersection with A as Hx_1, \dots, Hx_k , and set $A_{x_i} := A \cap Hx_i$. Suppose without loss of generality that $|A_{x_1}|$ is the largest or joint-largest of these sets. Of course it has cardinality at least $|A|/k$. Then $A^2 \cap H$ contains $A_{x_1}A_{x_1}^{-1}$, and hence also has cardinality at least $|A|/k$. Now the intersection of A^{n+1} with each coset Hx_i has size at least $|A^n \cap H|$, and so $|A^n \cap H| \leq |A^{n+1}|/k$. By Lemma 2.4 this is at most $K^{C_n}|A|/k$. Comparing these inequalities gives the result. \square

Our final combinatorial preliminary is of a rather different type. It is a beautiful “sum-product” result of Solymosi [14].

Lemma 2.8 (Sum-product over \mathbb{C}). *Suppose that U, V, W are any three finite subsets of \mathbb{C} . Then we have*

$$|U + V||UW| \geq c|U|^{3/2}|V|^{1/2}|W|^{1/2}.$$

The use of sum-product phenomena in the study of approximate groups of matrices is very natural, since matrix multiplication involves both addition and multiplication of the entries. This observation has been exploited before in the work of Helfgott [8, 9] and Chang [4, 5].

3. PROOF OF THE MAIN THEOREMS

By Lemma 2.3 and Mal’cev’s theorem it suffices to establish Theorem 1.4 for approximate subgroups of $\text{Upp}_n(\mathbb{C})$. The argument proceeds by induction on n , and for

the purposes of establishing the inductive step it turns out to be advantageous to work with the following (slightly more precise) variant of Theorem 1.4.

Theorem 1.4'. Let $K \geq 1$. Suppose that $G = \text{Upp}_n(\mathbb{C})$ and that $A \subseteq G$ is a set with $|A^3| \leq K|A|$. Then there is some set $A' \subseteq A$ with $|A'| \geq K^{-C}|A|$ which is contained in a left coset of a nilpotent subgroup of $\text{Upp}_n(\mathbb{C})$ of step at most $n - 1$.

Deduction of Theorem 1.4 for approximate subgroups of $\text{Upp}_n(\mathbb{C})$. Let $A \subseteq \text{Upp}_n(\mathbb{C})$ be a K -approximate group. Then by Proposition 2.1 (i) we have $|A^3| \leq K^2|A|$, and so we may apply Theorem 1.4' (with K replaced by K^2). It follows that there is some set $A' \subseteq A$ with $|A'| \geq K^{-C}|A|$ which is contained in xH , some left coset of a nilpotent group $H \leq \text{Upp}_n(\mathbb{C})$ of step at most $n - 1$. The set $S = A'^{-1}A'$ then generates a nilpotent subgroup of $\text{Upp}_n(\mathbb{C})$, and by Lemmas 2.2 and 2.4 the set S^2 is symmetric and K^C -controls A . By the second sentence of Proposition 2.1 (v) and Lemma 2.4 the set S^6 is a K^C -approximate group which K^C -controls A . This concludes the proof of Theorem 1.4. \square

It remains, then, to establish Theorem 1.4'. As we said this proceeds by induction, the case $n = 1$ being trivial since $\text{Upp}_1(\mathbb{C}) \cong \mathbb{C}$ is abelian. The key to the inductive step is the observation that there are two different homomorphisms $\pi, \pi' : \text{Upp}_n(\mathbb{C}) \rightarrow \text{Upp}_{n-1}(\mathbb{C})$ defined by

$$\pi((x_{ij})_{1 \leq i \leq j \leq n}) = (x_{ij})_{1 \leq i \leq j \leq n-1} \quad \text{and} \quad \pi'((x_{ij})_{1 \leq i \leq j \leq n}) = (x_{i+1,j+1})_{1 \leq i \leq j \leq n-1}.$$

Let us take a set $A \subseteq \text{Upp}_n(\mathbb{C})$ with $|A^3| \leq K|A|$ and apply the two homomorphisms π and π' to it. By Lemma 2.6 we have $|\pi(A)^3| \leq K^4|\pi(A)|$ and hence by inductive hypothesis there is some set $A' \subseteq A$ with $|\pi(A')| \geq K^{-C}|\pi(A)|$ such that $\pi(A')$ is contained in a left coset of a nilpotent subgroup of $\text{Upp}_{n-1}(\mathbb{C})$. It follows from Lemma 2.5 that $|A'| \geq K^{-C}|A|$. We now apply π' to *this* set, obtaining a further subset $A'' \subseteq A'$ with $|A''| \geq K^{-C}|A|$ and such that $\pi'(A'')$ is contained in a left coset of a nilpotent subgroup of $\text{Upp}_{n-1}(\mathbb{C})$. For ease of notation we drop the double dash and assume that both $\pi(A')$ and $\pi'(A')$ lie in left cosets of nilpotent subgroups of $\text{Upp}_{n-1}(\mathbb{C})$ of step at most $n - 1$. The sets $\pi(A'A'^{-1})$ and $\pi'(A'A'^{-1})$ are then contained in nilpotent subgroups of $\text{Upp}_{n-1}(\mathbb{C})$ of step at most $n - 1$. Write $B := A'A'^{-1}$; thus B is symmetric and by Lemma 2.4 we have

$$|B^3| \leq |A^6| \leq K^C|A| \leq K^{C_d}|B|. \quad (3.1)$$

Our conclusion implies that for any $b_1, \dots, b_n \in B$ the commutator

$$[b_1, [b_2, [b_3, \dots [b_{n-1}, b_n] \dots]]]$$

lies in $\ker \pi \cap \ker \pi'$, which is none other than the normal subgroup $H \triangleleft \text{Upp}_n(\mathbb{C})$ consisting of matrices of the form $m_\lambda = \begin{pmatrix} 1 & 0 & \cdots & \lambda \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$, $\lambda \in \mathbb{C}$. Note that $m_\lambda m_\mu = m_{\lambda+\mu}$.

We will make use of the following well-known lemma:

Lemma 3.1. *Suppose that G is a group generated by finitely many elements x_1, \dots, x_k , and that every $(s+1)$ -fold nested commutator $[x_{i_1}, [x_{i_2}, [x_{i_3}, \dots [x_{i_s}, x_{i_{s+1}}] \dots]]]$ is equal to the identity. Then G is s -step nilpotent.*

Proof. This is [12, Lemma 8.17]. □

Let $N := 3 \cdot 2^{n-1} - 2$, the length of the commutator $[b_1, [b_2, [b_3, \dots [b_{n-1}, b_n] \dots]]]$ as a word in the b_i . Now if $B^N \cap H$ contains only the identity matrix then every n -fold nested commutator $[b_1, [b_2, [b_3, \dots [b_{n-1}, b_n] \dots]]]$, where $b_1, \dots, b_n \in B$, is equal to the identity. By Lemma 3.1 it follows that B generates an $(n-1)$ -step nilpotent group.

Otherwise, $B^N \cap H$ contains at least one matrix m_λ with $\lambda \neq 0$. Note that B^N also contains the identity matrix. We now divide into two cases.

Let $D = K^{\gamma_n}$, where $\gamma_n > 0$ is to be specified later. Suppose first that there are at least D different ratios $T = \{t_1, \dots, t_D\}$ amongst the values of x_{11}/x_{nn} occurring in B^N . For each $k = 1, \dots, D$ choose some $y_k = (x_{ij}^{(k)})_{1 \leq i \leq j \leq n} \in B^N$ with $x_{11}^{(k)}/x_{nn}^{(k)} = t_k$. An easy computation gives $y_k m_\lambda y_k^{-1} = m_{t_k \lambda}$. Thus if the set $S \subseteq \mathbb{R}$ is defined by

$$B^N \cap H = \{m_\lambda : \lambda \in S\}$$

then $B^{3N} \cap H$ contains the elements $\{m_\lambda : \lambda \in TS\}$ and also the elements $\{m_\lambda : \lambda \in S + S\}$. However by Solymosi's sum-product bound (Lemma 2.8) we have

$$|S + S| \cdot |ST| \geq c|S||T|^{1/2}.$$

Thus

$$|B^{3N} \cap H| \geq c\sqrt{D}|B^N \cap H|.$$

If γ_n is sufficiently large then, together with Lemma 3.1, this contradicts Lemma 2.7.

Such a set T cannot, therefore, exist. *A fortiori* there are no more than D different ratios x_{11}/x_{nn} occurring amongst the elements (x_{ij}) of A' , and so there is a set $C \subseteq A'A'^{-1} = B$, of size at least $|B|/D \gg K^{-C_n}|A|$, with the property that $x_{11}/x_{nn} = 1$ for all $(x_{ij}) \in C$. Then every element of C commutes with H and so

$$[c, [b_1, [b_2, \dots [b_{n-1}, b_n] \dots]]] = \text{id}$$

whenever $b_1, \dots, b_n \in B$ and $c \in C$, and hence in particular when all of b_1, \dots, b_n, c lie in C . It follows from Lemma 3.1 that C is contained in a nilpotent subgroup of $\text{GL}_n(\mathbb{C})$.

We have examined two cases, and in both of them have located a set $X \subseteq A'A'^{-1}$ with $|X| \geq K^{-C_n}|A|$ which is contained in a nilpotent subgroup of $\text{Upp}_n(\mathbb{C})$ of step at most

n . Remember what it is that we are trying to prove: that a substantial fraction of A lies inside some *coset* of a nilpotent subgroup of $\mathrm{GL}_n(\mathbb{C})$. To establish this it suffices to show that there is $x \in \mathrm{GL}_n(\mathbb{C})$ such that $|A \cap xC| \gg K^{-O_n(1)}|A|$. Write $r(x) = |A \cap xC|$, which is also the number of representations of x as ac^{-1} with $a \in A, c \in C$. Then

$$\sum_x r(x) = |A||C| \gg K^{-O_n(1)}|A|^2.$$

However $r(x) = 0$ unless $x \in AC^{-1}$, a set which is contained in A^3 and hence has cardinality at most $K|A|$. It follows that there is at least one x for which $r(x) \gg K^{-O_n(1)}|A|$, as required. This completes the proof of the inductive step in Theorem 1.4', and hence the proof of Theorem 1.4. \square

It remains to deduce Corollary 1.5. To do this we need the following proposition concerning the structure of nilpotent subgroups of $\mathrm{Upp}_n(\mathbb{C})$. Here $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ is the n -dimensional torus.

Proposition 3.2. *Suppose that G is a nilpotent subgroup of $\mathrm{Upp}_n(\mathbb{C})$. Then G embeds into $\mathbb{T}^n \times \Gamma$, where Γ is a simply-connected nilpotent Lie group of dimension at most that of $\mathrm{Upp}_n(\mathbb{C})$ (and in particular is torsion-free).*

Although this proposition is certainly well-known to experts, we do not know of a particularly convenient source for it. The account which follows assumes a certain amount of background in the theory of complex algebraic groups such as may be found in the book of Humphreys [10]. A more self-contained discussion based on the treatment of Wehrfritz [19] may be found on the second author's website [6].

Proof. As G embeds in its Zariski-closure, and the Zariski-closure of a nilpotent group is still nilpotent, there is no loss of generality in assuming that G is Zariski-closed. We shall show that G is a direct product of two Zariski-closed subgroups G_s and G_u where G_s is isomorphic to a subgroup of $(\mathbb{C}^\times)^n$ and G_u is connected, simply connected and nilpotent. This is enough to prove the proposition, since $(\mathbb{C}^\times)^n$ is $\mathbb{T}^n \times \mathbb{R}^n$ as a real Lie group. When G is Zariski-connected, this assertion is Proposition 19.2 in Humphreys' book [10]. We recall the argument and explain why it continues to hold in the non-connected case.

Let G_u be the set of unipotent elements in G . Clearly $G_u = G \cap \mathrm{Uni}(\mathbb{C})$, where $\mathrm{Uni}(\mathbb{C})$ is the subgroup of $\mathrm{Upp}(\mathbb{C})$ consisting of matrices with all eigenvalues equal to 1. First we remark that G_u is connected (in either the topological or Zariski- sense, these being equivalent for complex algebraic groups). To see this note that if $\{g^n\}_{n \in \mathbb{Z}}$ lies in G_u so does $\{g^t\}_{t \in \mathbb{C}}$, since g^n is a polynomial function of n , and so G_u is path-connected.

Now it is known (see for example [2]) that the exponential map on a connected, simply-connected Lie group H such as $\text{Uni}(\mathbb{C})$ is a diffeomorphism and that it induces a correspondence between closed connected subgroups of H and Lie subalgebras of $\mathfrak{h} := \log H$. These last are certainly vector subspaces of \mathfrak{h} and hence are simply-connected, and hence so are all closed connected subgroups of H . In particular, G_u is simply-connected.

Let G_s be the set of semisimple elements in G . We claim that G_s is a subgroup and that it commutes pointwise with G_u . Once this is established it follows that $G = G_s \times G_u$, and furthermore the natural map $\text{Upp}(\mathbb{C}) \rightarrow (\mathbb{C}^\times)^n$ restricts to an isomorphism on G_s , thereby concluding the proof of the proposition. In fact it is enough to show that G_s commutes pointwise with G_u . To see this, let $x, y \in G_s$ be arbitrary and consider the commutator $z := xyx^{-1}y^{-1}$, which lies in G_u . Then we have

$$xyx^{-1} = z \cdot y = y \cdot z$$

and

$$xyx^{-1} = \text{id} \cdot (xyx^{-1}) = (xyx^{-1}) \cdot \text{id},$$

two decompositions of xyx^{-1} into commuting semisimple and unipotent parts. By the uniqueness of Jordan decomposition it follows that $xyx^{-1} = y$, and so $xy = yx$. But the product of two commuting semisimple transformations is semisimple, whence $xy \in G_s$.

To confirm the claim, then, it remains to establish that G_s commutes with G_u . To this end we quote [10, Theorem 18.3 (c)], which tells us that for each fixed $y \in G_s$ the map $\phi_y(x) := xyx^{-1}y^{-1}$ is a bijection from $\phi_y(G_u)$ to itself. Thus we can write each element of $\phi_y(G_u)$ as a commutator of arbitrarily high order involving elements of G . Since G is nilpotent, we are forced to conclude that $\phi_y(G_u) = \{\text{id}\}$ and hence that G_s does indeed commute with G_u . \square

Deduction of Corollary 1.5. The argument is very close in spirit to the “rectification principle” developed in [1]. Let A be a K -approximate subgroup of some solvable subgroup of $\text{GL}_n(\mathbb{C})$. By Theorem 1.4 there is a K^{C_n} -approximate subgroup $B \subseteq \langle A \rangle$ generating a nilpotent subgroup of $\text{Upp}_n(\mathbb{C})$ which K^{C_n} -controls A . By Proposition 3.2 this nilpotent subgroup embeds into $\mathbb{T}^n \times \Gamma$, where Γ is a simply-connected nilpotent Lie group. It therefore suffices to show that if $S \subseteq \mathbb{T}^n \times \Gamma$ is a K -approximate group then it is $e^{K^{C_n}}$ -controlled by a nilpotent progression in $\langle S \rangle$. By a trivial pigeon-hole argument there is a set $S' \subseteq S$, $|S'| \geq 6^{-n}|S|$, which is entirely contained in some box $\mathbf{x} + [-\frac{1}{12}, \frac{1}{12}]^n \subseteq \mathbb{T}^n$. Let $S'' = S'S'^{-1}$. We have $S'' \subseteq S^2$, $|S''| \geq 6^{-n}|S|$ and $S'' \subseteq [-\frac{1}{6}, \frac{1}{6}]^n \subseteq \mathbb{T}^n$. Consider the obvious lift $\psi : S'' \rightarrow [-\frac{1}{6}, \frac{1}{6}]^n \times \Gamma \subseteq \mathbb{R}^n \times \Gamma$ to the universal cover. It is clear that $|\psi(S'')^{\pm 3}| = |S''^{\pm 3}|$, and so by Proposition 2.1 $\psi(S'')^{\pm 3}$ is a K^{C_n} -approximate group which K^{C_n} -controls $\psi(S'')$. By the main result of [3] it follows that there is a nilpotent progression $P \subseteq \langle \psi(S'') \rangle$ of dimension at most

K^{C_n} which $e^{K^{C_n}}$ -controls $\psi(S'')^{\pm 3}$. The image of this progression under the canonical projection back down to $\mathbb{T}^n \times \Gamma$ is a nilpotent progression Q of the same dimension which is contained in $\langle S \rangle$, and which $e^{K^{C_n}}$ -controls the set $S''^{\pm 3}$. But by Proposition 2.1 (v) $S''^{\pm 3}$ is a K^{C_n} -approximate group which K^{C_n} -controls S . Hence S is $e^{K^{C_n}}$ -controlled by Q . \square

REFERENCES

- [1] Y. Bilu, V. Lev and I. Z. Ruzsa, *Rectification principles in additive number theory*, Discr. Comput. Geo. **19** (1998), 343–353.
- [2] N. Bourbaki, *Groupes et algèbres de Lie*, vol I.
- [3] E. Breuillard and B. J. Green, *Approximate subgroups of nilpotent Lie groups*, preprint available at [arXiv:0906.3598](https://arxiv.org/abs/0906.3598).
- [4] M-C. Chang, *Product theorems in SL_2 and SL_3* , J. Math. Jussieu, **7**(1), 1–25, (2008).
- [5] ———, *Convolution of discrete measures on linear groups*, preprint.
- [6] B. J. Green, *On nilpotent linear groups*, notes available at <http://www.dpmms.cam.ac.uk/~bjg23/notes.html>
- [7] M. Hall, *The theory of groups*, Chelsea Publishing Co., New York, 1976.
- [8] H. A. Helfgott, *Growth and generation in $SL_2(\mathbb{Z}/p\mathbb{Z})$* , Ann. of Math. (2) **167** (2008), no. 2, 601–623.
- [9] H. A. Helfgott, *Growth in $SL_3(\mathbb{Z}/p\mathbb{Z})$* , preprint available at [arXiv:0807.2027](https://arxiv.org/abs/0807.2027).
- [10] J.E. Humphreys, *Linear algebraic groups*, Springer-Verlag, New-York, 1975.
- [11] A. I. Mal'cev, *On certain classes of infinite soluble groups*, (Russian) Mat. Sb. **28** (1951), 567–588, transl. in Amer. Math. Soc. Transl. (2) **2** (1956), 1–21.
- [12] M.S. Raghunathan, *Discrete Subgroups of Lie Groups*, Springer-Verlag, New York, 1972.
- [13] T. Sanders, *From polynomial growth to metric ball in monomial groups*, preprint.
- [14] J. Solymosi, *On sumsets and product sets of complex numbers*, J. Théorie des nombres de Bordeaux **17** (2005), 921–924.
- [15] D. A. Suprunenko and R. I. Tyškevič, *Reducible locally nilpotent linear groups* (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **24** (1960), 787–806.
- [16] T. C. Tao, *Product-set estimates in noncommutative groups*, Combinatorica **28** (2008), no. 5, 547–594.
- [17] ———, *Freiman's theorem in solvable groups*, preprint available at [arXiv:0906.3535](https://arxiv.org/abs/0906.3535).
- [18] T. C. Tao and V. H. Vu, *Additive Combinatorics*, CUP 2006.
- [19] B. A. F. Wehrfritz, *Infinite linear groups*, Queen Mary College Mathematics Notes 1969.

LABORATOIRE DE MATHÉMATIQUES, BÂTIMENT 425, UNIVERSITÉ PARIS-SUD, 91405 ORSAY, FRANCE.

E-mail address: emmanuel.breuillard@math.u-psud.fr

CENTRE FOR MATHEMATICAL SCIENCES, WILBERFORCE ROAD, CAMBRIDGE CB3 0WA, ENGLAND

E-mail address: b.j.green@dpmms.cam.ac.uk